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A NOTE ON THE BOUNDARY CONDITIONS OF TOUPIN'S STRAIN-GRADIENT THEORY

JEFFREY L. BLEUSTEIN
Department of Engineering and Applied Science,
Yale University, New Haven, Connecticut

Abstract—In this paper the relationship between the boundary conditions of the linear form of Toupin's strain-gradient theory and the boundary conditions of Mindlin's micro-structure theory is established. This relationship provides a clear physical interpretation for the external forces in the strain-gradient theory.

INTRODUCTION

It is well known, see e.g. [1–3], that certain indeterminacies exist in theories of elasticity which incorporate the second or higher gradients of the displacement in addition to the strain in the expression for the potential energy. These indeterminacies lead, in turn, to complicated forms for the boundary conditions, forms which are difficult to interpret physically. It is also well known, see e.g. [3–5], that these indeterminacies can be removed by constructing related theories which admit additional independent kinematic variables. In these more general theories the boundary conditions have a simple physical interpretation.

In the present paper, it is shown how the boundary conditions of a fully determinate theory, viz. Mindlin's linear theory of an elastic continuum with micro-structure [3], can be reduced to those of a corresponding indeterminate theory, viz the linear form ([3], Section 10) of Toupin's strain gradient theory [1], thus providing a clear physical interpretation for the boundary conditions of Toupin's theory.

THE CONTINUUM WITH MICRO-STRUCTURE

In [3], Mindlin derived a linear theory for an elastic continuum with a deformable micro-structure. A rectangular cartesian system with coordinates $x_i, i = 1, 2, 3$, was employed, and the kinematic quantities

$$\begin{aligned} \varepsilon_{ij} &\equiv \tfrac{1}{2}(\partial_i u_j + \partial_j u_i), \text{ the macro-strain,} \\ \gamma_{ij} &\equiv \partial_i u_j - \psi_{ij}, \text{ the relative deformation,} \qquad (1) \\ \varkappa_{ijk} &\equiv \partial_i \psi_{jk}, \text{ the micro-deformation gradient,} \end{aligned}$$

were introduced. In equation (1) u_i is the macro-displacement field, ψ_{ij} is the micro-deformation field and $\partial_i (\equiv \partial/\partial x_i)$ is the gradient operator. For the equilibrium case, the field equations and boundary conditions are contained in the variational equation

$$\delta \mathscr{W} = \delta \mathscr{W}_1 \tag{2}$$

where $\delta\mathscr{W}$, defined in terms of $W(\varepsilon_{ij}, \gamma_{ij}, \varkappa_{ijk})$ (the potential energy per unit volume) by

$$\delta \mathscr{W} \equiv \int_V \delta W(\varepsilon_{ij}, \gamma_{ij}, \varkappa_{ijk})\, \mathrm{d}V, \tag{3}$$

is the variation of the total potential energy, and $\delta\mathscr{W}_1$ defined by

$$\delta\mathscr{W}_1 \equiv \int_V f_j \delta u_j \,\mathrm{d}V + \int_V \Phi_{jk}\delta\psi_{jk}\,\mathrm{d}V + \int_S t_j\delta u_j\,\mathrm{d}S + \int_S T_{jk}\delta\psi_{jk}\,\mathrm{d}S \tag{4}$$

is the variation of the work done by the external forces on the material contained in the volume V bounded by the surface S. In equations (2), (3) and (4) the twelve quantities δu_j and $\delta\psi_{jk}$ are varied independently. From the definitions of u_i and ψ_{ij} and the definition of $\delta\mathscr{W}_1$, Mindlin was able to discuss the physical significances of the quantities f_j, Φ_{jk}, t_j and T_{jk}, i.e. f_j is the body force per unit volume, Φ_{jk} is the body double force per unit volume, t_j is the surface force per unit area (traction) and T_{jk} is the surface double force per unit area (or double traction). In obtaining solutions to boundary value problems within the framework of the micro-structure theory it is evident that the twelve quantities f_j and Φ_{jk} are to be specified at every point in V and that twelve quantities, e.g. t_i and T_{jk} are to be specified at every point on S.

In the same paper, Mindlin showed how the potential energy of the continuum with micro-structure could be reduced to the potential energy of a micro-homogeneous continuum. Then, performing the variation of the total potential energy for independent variations δu_j, the linear equations of Toupin's strain-gradient theory were recovered. In [3], Mindlin took the form of the expression for the variation of the potential energy as motivation for the adoption of a form for the variation of the work done by external forces. In the present paper, we take as our starting point the expression, equation (4), for the variation of the work done by the external forces in a continuum with micro-structure. Then, passing to the case of a micro-homogeneous continuum we obtain a new interpretation for the boundary conditions of Toupin's theory.

REDUCTION TO THE STRAIN-GRADIENT THEORY

In [3], Section 10, Mindlin has shown that the theory of a micro-homogeneous material can be obtained from the theory of a material with micro-structure by causing the micro-medium to merge with the macro-medium. Mathematically, this is expressed by the condition

$$\gamma_{ij} = 0 \tag{5}$$

so that, from equation (1),

$$\psi_{ij} \to \partial_i u_j, \qquad \varkappa_{ijk} \to \partial_i\partial_j u_k \equiv \mathring{\varkappa}_{ijk}. \tag{6}$$

In view of equation (5), (6), the variation of the total potential energy

$$\delta\mathscr{W} \to \delta\mathring{\mathscr{W}} \equiv \int_V \delta\mathring{\mathscr{W}}(\varepsilon_{ij}, \mathring{\varkappa}_{ijk})\,\mathrm{d}V, \tag{7}$$

and the variation of the work done by the external forces

$$\delta\mathscr{W}_1 \to \delta\mathring{\mathscr{W}}_1 \equiv \int_V f_j\delta u_j\,\mathrm{d}V + \int_V \Phi_{jk}\partial_j(\delta u_k)\,\mathrm{d}V + \int_S t_j\delta u_j\,\mathrm{d}S + \int_S T_{jk}\partial_j(\delta u_k)\,\mathrm{d}S \tag{8}$$

Following Mindlin, we define the stresses

$$\mathring{\tau}_{ij} \equiv \frac{\partial\mathring{W}}{\partial\varepsilon_{ij}}, \qquad \mathring{\mu}_{ijk} \equiv \frac{\partial\mathring{W}}{\partial\mathring{\varkappa}_{ijk}}, \tag{9}$$

and by a process similar to that employed in [3], Section 9, we find that

$$
\begin{aligned}
\delta\mathring{\mathscr{W}} = & -\int_V \partial_j(\mathring{\tau}_{jk}-\partial_i\mathring{\mu}_{ijk})\delta u_k \,\mathrm{d}V \\
& +\int_S [n_j(\mathring{\tau}_{jk}-\partial_i\mathring{\mu}_{ijk})-D_j(n_i\mathring{\mu}_{ijk})+(D_l n_l)n_j n_i\mathring{\mu}_{ijk}]\delta u_k \,\mathrm{d}S \\
& +\int_S n_i n_j \mathring{\mu}_{ijk} D\delta u_k \,\mathrm{d}S \\
& +\oint_C [m_j n_i \mathring{\mu}_{ijk}]\delta u_k \,\mathrm{d}s.
\end{aligned} \tag{10}
$$

In equation (10), $D_j\,(\equiv(\delta_{jk}-n_j n_k)\partial_k)$ is the surface gradient operator, $D\,(\equiv n_k\partial_k)$ is the normal gradient operator, C is an edge formed by the intersection of two portions, S_1 and S_2 of the closed surface S, and the bold face brackets **[ ]** indicate that the enclosed quantity is the difference between the values on S_1 and S_2. Also, n_i is the unit outward normal to the surface, s_i is the unit tangent vector to the curve C and $m_k \equiv e_{ijk}s_i n_j$.

For the variation of the work done by the external forces, we note that

$$\Phi_{jk}\partial_j\delta u_k = \partial_j(\Phi_{jk}\delta u_k) - (\partial_j\Phi_{jk})\delta u_k. \tag{11}$$

An application of the divergence theorem then gives

$$\int_V \Phi_{jk}\partial_j\delta u_k \,\mathrm{d}V = \int_S n_j\Phi_{jk}\delta u_k \,\mathrm{d}S - \int_V (\partial_j\Phi_{jk})\delta u_k \,\mathrm{d}V. \tag{12}$$

Also, we can rewrite

$$
\begin{aligned}
T_{jk}\partial_j\delta u_k &= T_{jk}(D_j+n_j D)\delta u_k \\
&= D_j(T_{jk}\delta u_k)-(D_j T_{jk})\delta u_k+n_j T_{jk}D\delta u_k.
\end{aligned} \tag{13}
$$

Thus,

$$\int_S T_{jk}\partial_j\delta u_k \,\mathrm{d}S = \int_S D_j(T_{jk}\delta u_k)\,\mathrm{d}S - \int_S (D_j T_{jk})\delta u_k \,\mathrm{d}S + \int_S n_j T_{jk} D\delta u_k \,\mathrm{d}S. \tag{14}$$

Applying the surface divergence theorem ([6], p. 222) to the first integral on the right-hand side of equation (14) we find

$$\int_S D_j(T_{jk}\delta u_k)\,\mathrm{d}S = \int_S (D_i n_i)n_j T_{jk}\delta u_k \,\mathrm{d}S + \oint_C [m_j T_{jk}]\delta u_k \,\mathrm{d}s. \tag{15}$$

Inserting equations (12), (14) and (15) into (8) and combining terms we obtain

$$
\begin{aligned}
\delta\mathring{\mathscr{W}}_1 = & \int_V [f_k-\partial_j\Phi_{jk}]\delta u_k \,\mathrm{d}V \\
& +\int_S [t_k+n_j\Phi_{jk}+(D_i n_i)n_j T_{jk}-D_j T_{jk}]\delta u_k \,\mathrm{d}S \\
& +\int_S n_j T_{jk} D\delta u_k \,\mathrm{d}S \\
& +\oint_C [m_j T_{jk}]\delta u_k \,\mathrm{d}s.
\end{aligned} \tag{16}
$$

The variational equation

$$\delta \mathring{\mathscr{W}} = \delta \mathring{\mathscr{W}}_1 \tag{17}$$

then leads through equations (10) and (16) to the following stress-equations of motion and boundary conditions:

$$\begin{aligned} \partial_j(\mathring{\tau}_{jk} - \partial_i \mathring{\mu}_{ijk}) + f_k - \partial_j \Phi_{jk} &= 0 \qquad \text{in } V, \\ n_j(\mathring{\tau}_{jk} - \partial_i \mathring{\mu}_{ijk}) - D_j(n_i \mathring{\mu}_{ijk}) + (D_l n_l) n_j n_i \mathring{\mu}_{ijk} &= t_k + n_j \Phi_{jk} + (D_l n_l) n_j T_{jk} - D_j T_{jk} \qquad \text{on } S, \\ n_j n_i \mathring{\mu}_{ijk} &= n_j T_{jk} \qquad \text{on } S, \\ [m_j n_i \mathring{\mu}_{ijk}] &= [m_j T_{jk}] \qquad \text{on } C. \end{aligned} \tag{18}$$

The reader should compare equations (18) with Mindlin's equations (10.6). Equations (18) differ in that the acceleration terms have been dropped, the terms in $(10.6)_2$ have been regrouped, and explicit expressions in terms of the physically meaningful quantities f_j, Φ_{jk}, t_j and T_{jk} have been obtained for Mindlin's F_k, $\tilde{P}_k^\circ$, $\tilde{R}_k^\circ$ and $\tilde{E}_k^\circ$. The reader should also compare equations (18) with the appropriate linearized form of Toupin's equations (7.19) of [1]. In the present boundary conditions we *do not* make the restrictive assumption $n_k n_j T_{jk} = 0$ which would follow from Toupin's analysis. The boundary condition $(18)_2$ can be simplified by noting that the two terms which multiply $(D_l n_l)$ cancel if $(18)_3$ is satisfied. Thus, the stress equation of motion and the appropriate natural boundary conditions for the linear strain-gradient theory are

$$\begin{aligned} \partial_j(\mathring{\tau}_{jk} - \partial_i \mathring{\mu}_{ijk}) + f_k - \partial_j \Phi_{jk} &= 0 \qquad \text{in } V, \\ n_j(\mathring{\tau}_{jk} - \partial_i \mathring{\mu}_{ijk}) - D_j(n_i \mathring{\mu}_{ijk}) &= t_k + n_j \Phi_{jk} - D_j T_{jk} \qquad \text{on } S, \\ n_j n_i \mathring{\mu}_{ijk} &= n_j T_{jk} \qquad \text{on } S, \\ [m_j n_i \mathring{\mu}_{ijk}] &= [m_j T_{jk}] \qquad \text{on } C. \end{aligned} \tag{19}$$

The significance of equations (19) is the following: In a given application one would, presumably, have information about the quantities f_j, Φ_{jk}, throughout the volume and about the quantities t_j, T_{jk} on the surface. Equations (19) indicate the way in which this information is to be employed in setting up a boundary value problem within the framework of the linear strain-gradient theory.

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Résumé—Dans cet exposé les rapports entre les conditions des limites de la forme linéaire de la théorie relative au gradient de la tension de Toupin et aux conditions des limites de la théorie de microstructure de Mindlin sont établis. Ces rapports fournissent une interprétation physique claire pour les forces extérieures dans la théorie du gradient de la tension.

Zusammenfassung—Diese Arbeit seizt das Verhältnis zwischen den Grenzbedingungen der Linearform der Toupin'schen Spannungs-Gradienten Theorie und den Grenzbedingungen der Mindlin'schen Mikro Struktur Theorie fest. Das Verhältnis gibt eine klare physikalische Erklärung der ausseren Kräfte in der Spannungs-Gradienten Theorie.

Абстракт—В работе определяется зависимость между граничными условиями линейной формы теории градиента деформации Туиина и граничными условиями теории микроструктур Миндлина. Эта зависимость дает ясную физическую интерпретацию внешних сил в теории градиента деформации.